

Calculus III: Coursework 5
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The deadline is 4PM on Tuesday Nov 13th. Hand-in is to the green box on the ground floor of the Maths building.

1. Using Stokes's theorem evaluate $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$ over the surface S of the hemisphere defined by $x^2 + y^2 + z^2 = 4$, $z > 0$, where the z -component of $d\mathbf{S}$ is non-negative on S . [15]

Answer

By Stokes's theorem, the integral is equal to that of $\mathbf{F} \cdot d\mathbf{r}$ round the circle $x^2 + y^2 = 4$, $z = 0$, taken counterclockwise. Using $x = 2 \cos \theta$, $y = 2 \sin \theta$, we have

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} [(2 \cos \theta + 2 \sin \theta)(-2 \sin \theta) + (2 \sin \theta)(2 \cos \theta)] d\theta \\ &= - \int_0^{2\pi} 4 \sin^2 \theta d\theta = -4\pi \end{aligned}$$

Another way is to use Stokes's theorem to show that the integral over the hemisphere is the same as the one over the disk $x^2 + y^2 \leq 4$, $z = 0$, which has $d\mathbf{S} = \mathbf{k} dx dy$, $\nabla \times \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ and hence the integral is $\iint -dx dy$ over the disc, which is minus the area, hence -4π .

Marks A correct use of Stokes 4, parametrizing the circle 3, forming correct integral and limits 4, evaluation 4 (divide for partially correct work).

or: A correct use of Stokes 4, evaluation of curl 4, forming a correct integral over the disk 4, evaluation 3

Note there are many correct alternative versions

2. For each of the following fields $\mathbf{F}$, evaluate $\nabla \times \mathbf{F}$ and either find the general solution Φ satisfying $\mathbf{F} = \nabla \Phi$ everywhere, or show that no such Φ exists: [28]

- (a) $\mathbf{F} = z\mathbf{i} + x\mathbf{k}$
- (b) $\mathbf{F} = yz\mathbf{j} + zx\mathbf{k}$
- (c) $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} + (y + 1)\mathbf{k}$
- (d) $\mathbf{F} = 2y\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$.

Answer In each case, the curl can be calculated directly from the standard formulae, or if you spot that the $\mathbf{F}$ is conservative, you can find Φ and then just say $\nabla \times \mathbf{F} = \nabla \times \nabla \Phi = \mathbf{0}$.

For the conservative fields, Φ can be found by successive integration of components or from the line-integral definition (too many variants to write out!)

- (a) $\Phi = xz[+\text{constant}]$ and $\nabla \times \mathbf{F} = \mathbf{0}$.
- (b) $\nabla \times \mathbf{F} = -y\mathbf{i} - z\mathbf{j}$. No Φ is possible.
- (c) $\Phi = xy + yz + z[+\text{constant}]$ and $\nabla \times \mathbf{F} = \mathbf{0}$.
- (d) $\nabla \times \mathbf{F} = -2\mathbf{k}$. No Φ is possible.

Marks: in each part, 4 for the curl. 4 for Φ when it exists, 2 when it does not. No penalty for omitting the constants in Φ as the answer is still correct.

Mark scheme allows same reward for students who compute curls directly when it is not necessary and those who first show $\mathbf{F}$ is a gradient.

3. Write the following in the usual vector notation. [18]

- (i) $\delta_{ij}b_j$, (ii) $\epsilon_{ijk}a_kb_j$, (iii) δ_{kk} , (iv) c_md_m , (v) $\partial f/\partial x_j$,
- (vi) $c_ka_pc_mb_kd_p - \delta_{mn}a_n = \epsilon_{mkl}f_kd_l$.

Answers

- (i) $\mathbf{b}$
- (ii) $\mathbf{b} \times \mathbf{a}$
- (iii) 3
- (i) $\mathbf{c} \cdot \mathbf{d}$
- (v) ∇f
- (vi) $(\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})\mathbf{c} - \mathbf{a} = \mathbf{f} \times \mathbf{d}$

Marks 2 each for first 5 parts (all seen in lectures). 8 for last part (not seen), of which 4 for spotting the disguised dot products.

[Note: hard copies had the last expression misprinted as $c_ka_pc_mb_kd_p - \delta_{mn}a_n = \epsilon_{mkl}f_kd_l$. Be generous to students who take this form and make sensible remarks. I did announce the error on the website and in a lecture.]

4. Write the vector equation [7]

$$\mathbf{a} = (\mathbf{f} \cdot \mathbf{b})\mathbf{g} - \mathbf{c} \times \mathbf{d}$$

in index notation.

Answer $a_i = f_mb_mg_i - \epsilon_{ijk}c_jd_k$ (or equivalent)

Marks: 3 for any expression with a set of indices obeying the conventions, 2 for accuracy for each expression on the right (This question would have more marks were the terms not so close to parts of the previous question.)

5. Using index notation: [32]

- (a) Find an expression not involving cross-products for $\mathbf{a} \times (\mathbf{c} \times \mathbf{d})$;
- (b) Prove that

$$\nabla \cdot (\mathbf{r} \times \mathbf{b}) = 0$$

and

$$\nabla \times (\mathbf{r} \times \mathbf{b}) = -2\mathbf{b}$$

(c) Prove that

$$\nabla \cdot (\Phi \mathbf{F}) = \Phi(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla \Phi .$$

(d) Evaluate $\nabla(\mathbf{b} \cdot \mathbf{r})$ and $\nabla \times (\mathbf{a}(\mathbf{r} \cdot \mathbf{b}))$, where $\mathbf{a}$ and $\mathbf{b}$ are fixed (constant) vectors. [You may like to compare the effort involved with the first question in exercises 4.]

[Where appropriate you may use (without proof) the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

Answer

$$(a) \epsilon_{ijk}a_j\epsilon_{klm}c_ld_m = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im})a_jc_ld_m = a_m c_i d_m - a_l c_i d_i$$

$$\text{so } \mathbf{a} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{d}$$

$$(b) \partial_i(\epsilon_{ijk}x_jb_k) = \epsilon_{ijk}\delta_{ij}b_k = 0$$

(using “symmetric times skew is zero”).

$$\epsilon_{lmi}\partial_m(\epsilon_{ijk}x_jb_k) = \epsilon_{lmi}\epsilon_{ijk}\delta_{mj}b_k = \epsilon_{lmi}\epsilon_{imk}b_k = -2\delta_{lk}b_k = -2b_l.$$

$$(c) \partial_i(\Phi F_i) = \Phi \partial_i F_i + F_i \partial_i \Phi \text{ (by Leibnitz's Rule).}$$

$$(d) \partial_i(x_jb_j) = \delta_{ij}b_j = b_i \text{ so } \nabla(\mathbf{b} \cdot \mathbf{r}) = \mathbf{b}$$

$$\epsilon_{imk}\partial_m(a_k(x_jb_j)) = \epsilon_{imk}\delta_{jm}a_kb_j = \epsilon_{ijk}b_ja_k \text{ so } \nabla \times (\mathbf{a}(\mathbf{r} \cdot \mathbf{b})) = \mathbf{b} \times \mathbf{a}$$

Marks (a) 7 (3 for correct initial index expression) (b) 5 and 8 (again, 3 for initial index expression) (c) 4 (d) 3 and 5. The greater marks are for correct use of the double epsilon. This will a bit awkward to mark as students' choices of indices may vary quite a lot.

[Note: the hard copies have the second equation of (b) misprinted as

$$\nabla \times (\mathbf{r} \times \mathbf{b}) = 2\mathbf{b}.$$

This has also been announced. Students who used the wrong version should arrive at the answer that it was wrong – do not reward incorrect steps taken to get themselves to the incorrect answer! Correct steps get marks of course.]